Tracking Control of General Nonlinear Differential-Algebraic Equation Systems

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The problem of output tracking control is considered for a general class of nonlinear differential-algebraic systems. Regularization algorithm proposed here provides sufficient conditions for the existence of a regularizing feedback controller that renders the closed-loop system to have an index one. Based on the regularization algorithm, another algorithm is developed for constructing a tracking controller, which guarantees that the outputs of the closed-loop system track the desired signals asymptotically. Simulation results on a typical chemical process show that the developed design method gives a satisfactory control performance.

Introduction

Consider a system described by differential equations

$$\dot{x} = f(x,z,u) \tag{1}$$

algebraic equations

$$0 = g(x,z,u) \tag{2}$$

and outputs

$$y_i = h_i(x,z,u) \tag{3}$$

where $x \in R^n$, $z \in R^s$, $u \in R^m$, and $y = [y_1, \ldots, y_m]^T \in R^m$ are the vectors of state, algebraic, input, and output variables, respectively. Such a system is often called a differential-algebraic equation (DAE) system (also referred to as a singular, generalized, descriptor, or semistate system).

DAE systems arise naturally in many applications, such as chemical processes (Kumar and Daoutidis, 1995a,b), mechanical systems (You and Chen, 1993), and so on. DAE systems also arise from singular perturbation systems

$$\dot{x} = f(x,z,u) \tag{4}$$

$$\epsilon \dot{z} = g(x,z,u) \tag{5}$$

where $0 < \epsilon \ll 1$ is a small parameter. The slow subsystem obtained by letting $\epsilon = 0$ is the same as the DAE system described by Eqs. 1−2. So, the control theory for DAE systems may be used to design feedback controllers for singular perturbation systems (Christofides and Daoutidis, 1996). For many physical processes, the algebraic equations are implicit and singular in nature, inhibiting a direct elimination of such equations to obtain conventional ordinary differential equation (ODE) models. The application of existing control methods for ODE systems to processes with singular algebraic constraints clearly limits the controller performance and the control quality. These considerations motivate a need for developing a control method to deal with DAE systems directly.

It is known that not all initial conditions admit a smooth solution. It is assumed that the DAE system is solvable, that is, for a consistent set of initial conditions $(x(0),z(0))$ and smooth inputs $u(t)$, there exists locally a unique smooth solution $(x(t),z(t))$. The index v_d of a DAE system is the minimum number of times the algebraic equations have to be differentiated to obtain differential equations for the algebraic variables z (Brenan et al., 1989). Note that not all DAE systems have a well-defined index; especially, the index may not exist for arbitrary inputs when the constraints hidden behind the algebraic equations, which can be identified by differentiating the algebraic equations, depend on the inputs. Therefore, it is further assumed that the DAE system has a finite

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index υ_d. It is clearly seen that DAE system of Eqs. 1–2 has an index-one if the Jacobian matrix $\partial g/\partial z$ is nonsingular. For such systems, the algebraic equations can be solved theoretically for the algebraic variables z in terms of x and u to obtain an equivalent ODE representation.

For high-index DAE systems with $\partial q/\partial z$ singular, there exist some further constraint in x hidden in the algebraic Eq. 2. The differentiation of the algebraic equations can be repeatedly applied in order to identify all the hidden constraints, determine the algebraic variables, and obtain an equivalent ODE representation defined in a constrained state-space region. Generally speaking, the resulting ODE representation involves derivatives of input functions, which may degrade the control performance. On the other hand, the constrained state-space region may depend on the inputs. The dependence of the constrained state-space region on the inputs makes feedback control problems complicated, because the constrained state-space region depends on the (unknown) feedback control law for the inputs. In this article, we are concerned with the case where the constrained state-space region is independent of the inputs. A DAE system is called to be regular if it has a finite index and the constrained state-space region is independent of the inputs (Kumar and Daoutidis, 1995a). A large class of nonregular DAE systems can be made to be regular through feedback control. The problem of designing a feedback control law to render a nonregular DAE system regular is called a regularization problem, which will be discussed in the next section.

The numerical solution of DAE systems has been the subject of intense research activity in the past three decades. A great deal of progress has been made in the understanding of the mathematical structure of DAE systems. Several numerical techniques have been developed for the numerical solution of DAEs. Robust and efficient mathematical software is available for implementing the numerical methods (Brenan et al., 1989; Campbell and Griepentrog, 1995).

The majority of research on control of DAE systems has focused on linear systems (Campbell, 1982; Dai, 1989). Limited results have been available until recently on the control of nonlinear DAE systems, with few results on stabilization (McClamroch, 1990; Kumar and Daoutidis, 1995a,b; 1996), tracking (Krishnan and McClamroch, 1994; Liu, 1998a), H_∞ control (Wang et al., 1998), input-output decoupling (Liu and Čelikovský, 1997), disturbance decoupling (Liu, 1998b), and canonical form (Rouchon et al., 1992). A methodological framework for the feedback control of nonlinear DAE systems was introduced in Kumar and Daoutidis (1995a) and was further generalized in Kumar and Daoutidis (1995b) to address the control of DAE systems with disturbances. The developed methodology involved two steps: deriving a state-space realization of the constrained system and constructing a feedback controller on the basis of the derived state-space realization. This methodology has been extended in Kumar and Daoutidis (1996) to DAE systems for which the underlying constraints involve the inputs. Another method was developed in Liu and Čelikovský (1997) for the feedback control of nonlinear DAE systems. This method consists of two algorithms: a regularization algorithm and a decoupling algorithm. Instead of deriving the state space realization in Kumar and Daoutidis's method, the regularization algorithm identifies conditions for the existence of a regularizing feedback controller. Based on these conditions, the decoupling algorithm provides an approach for the construction of a decoupling controller without solving the algebraic variables. Such a design idea has been applied to solve the tracking problem (Liu, 1998a) and the disturbance decoupling problem (Liu, 1998b) for nonlinear DAE systems.

Both methods aforementioned are limited to DAE systems of the following affine form

$$\dot{x} = f_1(x) + p_1(x)z + g_1(x)u \tag{6}$$

$$0 = f_2(x) + p_2(x)z + g_2(x)u \tag{7}$$

However, many practical systems are modeled by Eqs. 1–2 instead of Eqs. 6–7 see, for example, the two-phase exothermic reactor and condenser (Kumar and Daoutidis, 1996), the reactive distillation column (Kumar and Daoutidis, 1999), the catalytic continuous stirred tank reactor (Christofides and Daoutidis, 1996), the liquid-liquid phase-transfer catalyzed reaction system (Chen et al., 1991), and so on. Motivated by this consideration, this work aims at a comprehensive framework for the controller synthesis for nonlinear DAE systems of the form of Eqs. 1–2.

Generally speaking, the design method developed by Kumar and Daoutidis (1995a,b; 1996) is not always suitable for control synthesis of DAE systems (Eqs. 1–2) because of difficulties in finding the equivalent ODE representation. Therefore, the design idea proposed by Liu and Čelikovský (1997) will be applied to construct a tracking controller so that the outputs of Eqs. 1–2 track the desired smooth signals $y^d(t)$ with bounded derivatives $\dot{y}^d(t)$, $\ddot{y}^d(t)$, $\cdots$. The design approach consists of two steps. The first step is to check if the given DAE system can be regularized by carrying out the regularization algorithm developed in the next section. The second step is to construct a tracking controller by performing the algorithm proposed later. Finally, the design method is tested by simulation on a chemical process composed of a two-phase reactor and condenser.

Regularization

This section is devoted to identifying conditions for the existence of a feedback controller, which guarantees that the closed-loop system is regular. Such a feedback controller is called a regularizing controller.

The following algorithm gives sufficient conditions for the existence of a regularizing controller, which can be considered as a generalization of both algorithms proposed by Kumar and Doautidis (1996) and Liu and Čelikovský (1997). The algorithm involves, in each step, (a) differentiation of the algebraic equations, and (b) elementary row operations on the algebraic equations produced in (a) to identify the underlying constraints in x, of which a minimal number involve $\dot{z}$ and $\dot{u}$.

Algorithm 1 (Regularization Algorithm)

Step 1: *Let $G^1(x,z,u) = g(x,z,u)$. Then the algebraic equations (Eq. 2) become*

$$0 = G^1(x,z,u) \tag{8}$$

Differentiating these algebraic equations with respect to time produces

$$0 = \frac{\partial G^1}{\partial x}\dot{x} + \frac{\partial G^1}{\partial z}\dot{z} + \frac{\partial G^1}{\partial u}\dot{u} \tag{9}$$

Set $p^1(x,z,u) = \text{rank}[(\partial G^1/\partial z)(\partial G^1/\partial u)]$. *Suppose* $p^1(x,z,u)$ *is constant in a neighborhood of* (x_0, z_0, u_0). *If* $p^1 = s$, *then let* $G_1^1 = G^1$ *and terminate the algorithm. Otherwise, rearrange the rows of* G^1 *so that first* p^1 *rows of the matrix* $[(\partial G^1/\partial z)(\partial G^1/\partial u)]$ *are linearly independent at* (x_0, z_0, u_0). *As a result,* G^1 *can be expressed as*

$$\begin{bmatrix} G_1^1 \\ G_2^1 \end{bmatrix}$$

and there exists a matrix $F_1^1(x,z,u)$ *such that*

$$\left[\frac{\partial G_2^1}{\partial z}\,\frac{\partial G_2^1}{\partial u}\right] = F_1^1(x,z,u)\left[\frac{\partial G_1^1}{\partial z}\,\frac{\partial G_1^1}{\partial u}\right] \tag{10}$$

Set $G^2(x,z,u) = [(\partial G_2^1/\partial x) - F_1^1(x,z,u)(\partial G_1^1/\partial x)]f(x,z,u)$. *Then, it is seen that Eq. 9 can be equivalently written as*

$$0 = \frac{\partial G_1^1}{\partial x}\dot{x} + \frac{\partial G_1^1}{\partial z}\dot{z} + \frac{\partial G_1^1}{\partial u}\dot{u} \tag{11}$$

$$0 = G^2(x,z,u) \tag{12}$$

which implies that some hidden constraints, say Eq. 12, have been found by differentiating the original algebraic equations. The next step is to differentiate the new algebraic equations (Eq. 12) in order to justify some constraints hidden in these algebraic equations.

Step *k: Suppose we have found hidden constraint functions* $G^2, \ldots, G^k$ *through steps 1 to* $k-1$. *This step is going to determine constraints behind the algebraic equations of the form*

$$0 = G^k(x,z,u) \tag{13}$$

Similar to the first step, differentiating these algebraic equations with respect to time produces

$$0 = \frac{\partial G^k}{\partial x}\dot{x} + \frac{\partial G^k}{\partial z}\dot{z} + \frac{\partial G^k}{\partial u}\dot{u} \tag{14}$$

Set

$$p^k(x,z,u) = \text{rank}\begin{bmatrix} \dfrac{\partial G_1^1}{\partial z} & \dfrac{\partial G_1^1}{\partial u} \\ \vdots & \vdots \\ \dfrac{\partial G_1^{k-1}}{\partial z} & \dfrac{\partial G_1^{k-1}}{\partial u} \\ \dfrac{\partial G^k}{\partial z} & \dfrac{\partial G^k}{\partial u} \end{bmatrix} \tag{15}$$

Suppose $p^k(x,z,u)$ *is constant in a neighborhood of* (x_0,z_0,u_0). *If* $p^k = s$, *then let* $G_1^k = G^k$ *and terminate the algorithm. Otherwise, rearrange the rows of* G^k *so that the first* p^k *rows of the matrix (Eq. 15) are linearly independent at* (x_0,z_0,u_0). *As a result,* G^k *can be partitioned as*

$$\begin{bmatrix} G_1^k \\ G_2^k \end{bmatrix}$$

and there exist matrices $F_j^k(x,z,u)$, $j = 1, \ldots, k$ *such that*

$$\left[\frac{\partial G_2^k}{\partial z}\,\frac{\partial G_2^k}{\partial u}\right] = \sum_{j=1}^{k} F_j^k(x,z,u)\left[\frac{\partial G_1^j}{\partial z}\,\frac{\partial G_1^j}{\partial u}\right] \tag{16}$$

Set

$$G^{k+1}(x,z,u) = \left(\frac{\partial G_2^k}{\partial x} - \sum_{j=1}^{k} F_j^k(x,z,u)\frac{\partial G_1^j}{\partial x}\right)f(x,z,u)$$

Then, it is easily seen that Eq. 14 can be equivalently expressed as

$$0 = \frac{\partial G_1^k}{\partial x}\dot{x} + \frac{\partial G_1^k}{\partial z}\dot{z} + \frac{\partial G_1^k}{\partial u}\dot{u} \tag{17}$$

$$0 = G^{k+1}(x,z,u) \tag{18}$$

Generally speaking, the regularization algorithm is not feasible because some of the constant assumptions may not be satisfied. The regularization algorithm is said to be feasible if the constant assumption at each step is satisfied. Throughout this article, only those systems for which the regularization algorithm is feasible will be considered, so the following assumption is made.

(A1) Algorithm 1 is feasible.

After carrying out the feasible regularization algorithm, we end up with a new set of algebraic equations

$$\begin{aligned} 0 &= G_1^1(x,z,u) \\ 0 &= G_2^1(x,z,u) \\ &\vdots \\ 0 &= G_1^{k^*-1}(x,z,u) \\ 0 &= G_2^{k^*-1}(x,z,u) \\ 0 &= G_1^{k^*}(x,z,u) \end{aligned} \tag{19}$$

which can be equivalently expressed as

$$\tilde{G}(x,z,u) = \begin{bmatrix} G_2^1(x,z,u) \\ \vdots \\ G_2^{k^*-1}(x,z,u) \end{bmatrix} = 0$$

$$G(x,z,u) = \begin{bmatrix} G_1^1(x,z,u) \\ \vdots \\ G_1^{k^*}(x,z,u) \end{bmatrix} = 0 \tag{20}$$

where

$$\operatorname{rank}\begin{bmatrix}\dfrac{\partial\tilde{G}}{\partial z} & \dfrac{\partial\tilde{G}}{\partial u}\\ \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u}\end{bmatrix} = s$$

Remark 1. *Suppose the DAE system (Eqs. 1−2) has an index υ_d. By differentiating the algebraic equations (Eq. 2) υ_d times, $\dot{z}$ can be theoretically determined as*

$$\dot{z} = Z(x,u,\dot{u},\ldots,u^{(u_d)}) \tag{21}$$

It is not difficult to induce that the feasible algorithm will be terminated at step k^*, which is bounded by $\upsilon_d - u_d + 1$ because the algebraic equations (Eq. 20) obtained after differentiating the algebraic equations (Eq. 2) $k^* - 1$ times has to be differentiated u^d times to uniquely determine $\dot{z}$, that is, $k^* - 1 + u^2 \le \upsilon^d$.

The following assumptions are made, which, together with (A1), provide sufficient conditions for the existence of a regularizing controller.

(A2) The matrix $\partial G/\partial z$ has constant rank s_1.

(A3) The matrix

$$\begin{bmatrix}\dfrac{\partial f}{\partial z}\\ \dfrac{\partial G}{\partial z}\end{bmatrix}$$

has constant full column rank s.

Remark 2 *(A3) is a necessary condition for the system (Eqs. 6−7) to be solvable and have a finite index for any smooth $u(t)$, see Lemma 1 in Kumar and Daoutidis (1996).*

In what follows, we will illustrate that, under the above assumption, the DAE system can be made to be regular by a state feedback controller.

For convenience in illustration and without loss of generality, reorder x, z, and u, partition them as

$$x = \begin{bmatrix}x_1\\x_2\end{bmatrix} \qquad z = \begin{bmatrix}z_1\\z_2\end{bmatrix} \qquad u = \begin{bmatrix}u_1\\u_2\end{bmatrix}$$

and decompose $f(x,z,u)$ and $G(x,z,u)$ compatibly with x and z, respectively

$$f(x,z,u) = \begin{bmatrix}f_1(x,z_1,z_2u_1,u_2)\\ f_2(x,z_1,z_2u_1,u_2)\end{bmatrix}$$

$$G(x,z,u) = \begin{bmatrix}G_1(x,z_1,z_2u_1,u_2)\\ G_2(x,z_1,z_2u_1,u_2)\end{bmatrix}$$

so that the matrices $\partial G_1/\partial z_1$ and

$$\begin{bmatrix}\dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2}\\ \dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial z_2}\end{bmatrix} \tag{22}$$

are nonsingular and the matrix

$$\begin{bmatrix}\dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial z_2} & \dfrac{\partial G_1}{\partial u_1}\\ \dfrac{\partial G_2}{\partial z_1} & \dfrac{\partial G_2}{\partial z_2} & \dfrac{\partial G_2}{\partial u_1}\end{bmatrix}$$

has full row rank, where $\dim(x_1) = s - s_1$, $\dim(z_1) = s_1$, $\dim(u_1) = s - s_1$, $\dim(G_1) = s_1$. Then, the dynamic feedback controller

$$\dot{w} = \upsilon_1 \tag{23}$$

$$G_2(x,z_1,z_2,u_1,u_2) = x_1 + w \tag{24}$$

$$u_2 = \upsilon_2 \tag{25}$$

makes the original system regular. As a matter of fact, the corresponding closed-loop system is given by

$$\dot{x}_1 = f_1(x,z_1,z_2,u_1,u_2) \tag{26}$$

$$\dot{x}_2 = f_2(x,z_1,z_2,u_1,u_2) \tag{27}$$

$$\dot{w} = \upsilon_1 \tag{28}$$

$$0 = G_1(x,z_1,z_2,u_1,u_2) \tag{29}$$

$$0 = x_1 + w \tag{30}$$

$$u_2 = \upsilon_2 \tag{31}$$

By differentiating Eq. 30, one gets

$$0 = f_1(x,z_1,z_2,u_1,u_2) + \upsilon_1 \tag{32}$$

The closed-loop system (Eqs. 26−32) is regular, which is verified in Appendix A.

Note that if the algorithm 1 is feasible and the matrix $\partial G/\partial z$ is nonsingular, then (A2)−(A3) are always true and the regularizing feedback (Eqs. 23−25) is not necessary, that is, the original system is regular.

Now let us look at the following DAE system

$$\dot{x}_1 = x_2 + z_1 + 2z_2 \tag{33}$$

$$\dot{x}_2 = u + x_1 + e^{x_1} - x_2 \tag{34}$$

$$\dot{x}_3 = u + e^{x_1} \tag{35}$$

$$0 = x_3 \tag{36}$$

$$0 = (z_1 + u)e^{z_1+u} + x_2 \tag{37}$$

$$y = x_1 \tag{38}$$

By carrying out Algorithm 1, we get the following two algebraic equations

$$0 = (z_1 + u)e^{z_1+u} + x_2 = G_1^1 \tag{39}$$

$$0 = u + e^{x_1} = G_2^1 = G_1^2 \tag{40}$$

It is easily seen that the index of the system is $v_d = 3$ and $u_d = 2$, and the algorithm terminates at step $k^* = 2$. The regularizing feedback can be chosen as

$$\dot{w} = v \tag{41}$$

$$u = w - e^{x_1} + x_1 \tag{42}$$

The original DAE system can be changed to an ODE system by a dynamic state feedback law. From the theoretical point of view, the existing control design techniques for ODE systems can be applied to the resulting state-space representation. However, such a design approach has the following two drawbacks. First, for some complex DAE systems, it is quite difficult, sometimes even impossible, to get equivalent state-space representation because of difficulties in determining z_1 and z_2 analytically. Secondly, control synthesis depends on the choice of regularizing controller. Therefore, in this article, the design idea developed by Liu and Čelikovský (1997) will be applied to construct an output tracking controller.

Feedback Design

After carrying out the feasible regularization algorithm, the original DAE system is changed to

$$\dot{x} = f(x,z,u) \tag{43}$$

$$0 = \tilde{G}(x,z,u) \tag{44}$$

$$0 = G(x,z,u) \tag{45}$$

$$y_i = h_i(x,z,u) \tag{46}$$

with $[(\partial G/\partial z)(\partial G/\partial u)]$ full row rank.

This section will propose another algorithm for the system (Eqs. 43–46), which is an extension of the second algorithm in Liu and Čelikovský (1997).

Algorithm 2

Step 1: *Let $H_i^1(x,z,u) = h_i(x,z,u)$. Then, the time derivative of H_i^1 is given by*

$$\dot{H}_i^1 = \frac{\partial H_i^1}{\partial x}\dot{x} + \frac{\partial H_i^1}{\partial z}\dot{z} + \frac{\partial H_i^1}{\partial u}\dot{u} \tag{47}$$

Define

$$q_i^1(x,z,u) = \text{rank}\begin{bmatrix} \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u} \\ \dfrac{\partial H_i^1}{\partial z} & \dfrac{\partial H_i^1}{\partial u} \end{bmatrix}$$

and assume $q_i^1(x,z,u)$ is constant in a neighborhood of x_0. If $q_i^1 = s + 1$, then define $r_i = 0$ and terminate the algorithm. Otherwise, there exists a matrix $E_i^1(x,z,u)$ so that

$$\begin{bmatrix} \dfrac{\partial H_i^1}{\partial z} & \dfrac{\partial H_i^1}{\partial u} \end{bmatrix} = E_i^1(x,z,u)\begin{bmatrix} \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u} \end{bmatrix} \tag{48}$$

Set $H_i^2(x,z,u) = [(\partial H_i^1/\partial x) - E_i^1(x,z,u)(\partial G/\partial x)]f(x,z,u)$. Then, Eq. 47 becomes

$$\dot{H}_i^1 = H_i^2(x,z,u) \tag{49}$$

where the equations

$$0 = \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial z}\dot{z} + \frac{\partial G}{\partial u}\dot{u}$$

have been applied.

Step k: *Suppose $H_i^2(x,z,u), \ldots, H_i^k(x,z,u)$ have been determined from Step 1 to Step $k-1$. Similar to the first step, differentiating H_i^k with respect to time gives*

$$\dot{H}_i^k = \frac{\partial H_i^k}{\partial x}\dot{x} + \frac{\partial H_i^k}{\partial z}\dot{z} + \frac{\partial H_i^k}{\partial u}\dot{u} \tag{50}$$

Define

$$q_i^k(x,z,u) = \text{rank}\begin{bmatrix} \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u} \\ \dfrac{\partial H_i^k}{\partial z} & \dfrac{\partial H_i^k}{\partial u} \end{bmatrix}$$

and assume $q_i^k(x,z,u)$ is a constant in a neighborhood of x_0. If $q_i^k = s + 1$, then define $r_i = k - 1$ and terminate the algorithm. Otherwise, there exists a matrix $E_i^k(x,z,u)$ so that

$$\begin{bmatrix} \dfrac{\partial H_i^k}{\partial z} & \dfrac{\partial H_i^k}{\partial u} \end{bmatrix} = E_i^k(x,z,u)\begin{bmatrix} \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u} \end{bmatrix} \tag{51}$$

Set $H_i^{k+1}(x,z,u) = [(\partial H_i^k/\partial x) - E_i^k(x,z,u)(\partial G/\partial x)]f(x,z,u)$. Then, Eq. 50 becomes as follows

$$\dot{H}_i^k = H_i^{k+1}(x,z,u) \tag{52}$$

Algorithm 2 terminates either at a finite step r_i, which is bounded by n, or never stops, which implies that the output y_i is independent of both z and u. If it does not terminate at a finite step, then define $r_i = n$. Performing Algorithm 2 on the ith output y_i gives

$$\begin{aligned} y_i &= H_i^1 \\ \dot{y}_i &= \dot{H}_i^1 = H_i^2 \\ &\vdots \\ y_{(r_i-1)} &= \dot{H}_i^{r_i-1} = H_i^{r_i} \\ y_i^{(r_i)} &= \dot{H}_i^{r_i} = H_i^{r_i+1}(x,z,u) \end{aligned} \tag{53}$$

Now we are ready to make the following assumption, which, together with (A1)–(A3), guarantee the existence of a tracking controller.

(**A4**): The matrix

$$\begin{bmatrix} \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial u} \\ \dfrac{\partial H_1^{r_1+1}}{\partial z} & \dfrac{\partial H_1^{r_1+1}}{\partial u} \\ \vdots & \vdots \\ \dfrac{\partial H_m^{r_m+1}}{\partial z} & \dfrac{\partial H_m^{r_m+1}}{\partial z} \end{bmatrix}$$

is nonsingular at x_0.

(**A4**) also means that r_i is the relative degree of the system (Isidori, 1995).

In order to design a feedback controller so that the output y_i tracks the desired signal $y_i^d(t)$, let us introduce the tracking error $e_i^1 = y_i - y_i^d(t)$. In addition, set $e_i^j = H_i^j - (y_i^d)^{(j-1)}(t)$ for $j = 1, \ldots, r_i$. Then, it follows from Eq. 53 that

$$\begin{aligned} \dot{e}_i^1 &= e_i^2 \\ &\vdots \\ \dot{e}_i^{r_i-1} &= e_i^{r_i} \\ \dot{e}_i^{r_i} &= H_i^{r_i+1}(x,z,u) - \left(y_i^d\right)^{(r_i)}(t) \end{aligned} \tag{54}$$

Now the introduction of the feedback controller composed of Eqs. 23–25 and

$$H_i^{r_i+1}(x,z,u) = -\sum_{j=1}^{r_i} d_i^j\left(H_i^j - \left(y_i^d\right)^{(j-1)}(t)\right) + \left(y_i^d\right)^{(r_i)}(t) \qquad i = 1, \ldots, m \tag{55}$$

to the system (Eq. 54) gives the following equations

$$\begin{aligned} \dot{e}_i^1 &= e_i^2 \\ &\vdots \\ \dot{e}_i^{r_i-1} &= e_i^{r_i} \\ \dot{e}_i^{r_i} &= -\sum_{j=1}^{r_i} d_i^j e_i^j \end{aligned} \tag{56}$$

which is asymptotically stable if the constants d_i^j are chosen in such a way that the polynomial $\lambda^{r_i} + d_i^{r_i}\lambda^{r_i-1} + \cdots + d_i^2\lambda + d_i^1$ is Hurwitz.

The process of designing a tracking feedback controller can be summarized as follows.

Step 1: Perform Algorithm 1 to determine $\tilde{G}(x,z,u)$ and $G(x,z,u)$;

Step 2: Perform Algorithm 2 to determine r_i and H_i^j for $j = 1, \ldots, r_i$, $i = 1, \ldots, m$.

Step 3: Choose d_i^j, $j = 1,\ldots,r_i$, $i = 1,\ldots,m$ so that $\lambda^{r_i} + d_i^{r_i}\lambda^{r_i-1} + \cdots + d_i^2\lambda + d_i^1$ is Hurwitz.

Step 4: The tracking controller is given by

$$\dot{w} = v_1 \tag{57}$$

$$G_2(x,z_1,z_2,u_1,u_2) = x_1 + w \tag{58}$$

$$u_2 = v_2 \tag{59}$$

$$H_i^{r_i+1}(x,z,u) = -\sum_{j=1}^{r_i} d_i^j\left(H_i^j - \left(y_i^d\right)^{(j-1)}(t)\right) + \left(y_i^d\right)^{(r_i)}(t), \qquad i = 1, \ldots, m \tag{60}$$

where z_1 and z_2 are determined by Eqs. 29 and 32.

Theoretically, the tracking controller for u_1 and u_2 can be uniquely determined from Eqs. 57–60, together with Eqs. 29 and 32 (see Appendix B). However, it is sometimes impossible to solve these equations analytically, so a numerical method is required.

Up to now, we have shown that the feedback control (Eqs. 57–60) achieves the output tracking. However, it does not guarantee the internal dynamics stability. The internal stability can be guaranteed by the following assumption.

(**A5**): The tracking dynamics are bounded input bounded state where the tracking dynamics is defined by Eq. 1 with constraints (Eq. 19) and $H_i^j = (y_i^d)^{(j-1)}(t)$ for $j = 1, \ldots, r_i$, $i = 1, \ldots, m$.

Remark 3. *Note that (A5) is stronger that the assumption of minimum-phase since (A5) means that the system is minimum-phase when $y_i^d(t) \equiv 0$. Theoretically, the tracking dynamics can be obtained by solving Eqs. 19 and $H_i^j = (y_i^d)^{(j-1)}(t)$ for $j = 1, \ldots, r_i$, $i = 1, \ldots, m$ for some components of x, z and u and substituting into Eq. 1. However, for those systems where it is impossible to find analytic solutions for x, z, and u, a numerical method has to be applied.*

Applying the design method above to the system (Eqs. 33–38) produces the following tracking controller

$$\dot{w} = v \tag{61}$$

$$u = w - e^{x_1} + x_1 \tag{62}$$

$$x_2 + z_1 + 2z_2 = -d\left[x_1 - y_d(t)\right] + \dot{y}_d(t) \tag{63}$$

where z_1 and z_2 are determined by

$$0 = x_2 + z_1 + 2z_2 + v \tag{64}$$

$$0 = (z_1 + u)e^{z_1+u} + x_2 \tag{65}$$

which has to be solved numerically. The corresponding closed-loop system is

$$\dot{x}_1 = -d\left[x_1 - y_d(t)\right] + \dot{y}_d(t) \tag{66}$$

$$\dot{x}_2 = x_1 - x_2 \tag{67}$$

$$y = x_1 \tag{68}$$

where the tracking dynamics is given by

$$\dot{x}_2 = -x_2 + y_d(t)$$

which is bounded-input and bounded-output.

Remark 4. *The main difference between the method presented in this article and the method developed by Kumar and Daoutidis (1996) is as follows: the tracking problem is addressed directly based on the original system in this article, whereas Ku-*

mar and Daoutidis addresses the tracking problem based on the ODE representation, which is obtained after applying a regularizing feedback control law. However, both control design methods give the same control performance (see next section).

Application of the Design Method

Consider the process comprised of a two-phase (vapor/liquid) reactor and a condenser. The process (Kumar and Daoutidis, 1996) can be described by 11 differential equations

$$\dot{M}_1^l = f_1 = F_B - F_{1l} + F_{2l} + N_{A1} - N_{C1} \tag{69}$$

$$\dot{x}_{A1} = f_2 = \frac{1}{M_1^l}\left[-F_B x_{A1} + F_{2l}(x_{A2} - x_{A1}) + N_{A1}(1 - x_{A1}) + N_{C1}x_{A1} - r_A\right] \tag{70}$$

$$\dot{x}_{B1} = f_3 = \frac{1}{M_1^l}\left[F_B(1 - x_{B1}) - F_{2l}x_{B1} - N_{A1}x_{B1} + N_{C1}x_{B1} - r_A\right] \tag{71}$$

$$\dot{M}_1^v = f_4 = F_A - F_{1v} - N_{A1} + N_{C1} \tag{72}$$

$$\dot{y}_{A1} = f_5 = \frac{1}{M_1^v}\left[F_A(1 - y_{A1}) - N_{A1}(1 - y_{A1}) - N_{C1}y_{A1}\right] \tag{73}$$

$$\dot{T}_1 = f_6 = \frac{1}{M_1^l + M_1^v}\left[F_A(T_A - T_1) + F_B(T_B - T_1) + F_{2l}(T_2 - T_1) + (N_{A1} - N_{C1})\frac{\Delta H^v}{c_p} - \frac{Q_1}{c_p} - r_A\frac{\Delta H_r}{c_p}\right] \tag{74}$$

$$\dot{M}_2^l = f_7 = N_{A2} + N_{C2} - F_{2l} \tag{75}$$

$$\dot{x}_{A2} = f_8 = \frac{1}{M_2^l}\left[N_{A2}(1 - x_{A2}) - N_{C2}x_{A2}\right] \tag{76}$$

$$\dot{M}_2^v = f_9 = F_{1v} - F_{2v} - N_{A2} - N_{C2} \tag{77}$$

$$\dot{y}_{A2} = f_{10} = \frac{1}{M_2^v}\left[F_{1v}(y_{A1} - y_{A2}) - N_{A2}(1 - y_{A2}) + N_{C2}y_{A2}\right] \tag{78}$$

$$\dot{T}_2 = f_{11} = \frac{1}{M_2^l + M_2^v}\left[F_{1v}(T_1 - T_2) + (N_{A2} + N_{C2})\frac{\Delta H^v}{c_p} - \frac{Q_2}{c_p}\right] \tag{79}$$

and 8 algebraic equations

$$0 = P_1 y_{A1} - P_{A1}^s x_{A1} \tag{80}$$

$$0 = P_1(1 - y_{A1}) - P_{C1}^s(1 - x_{A1} - x_{B1}) \tag{81}$$

Table 1. Process Variables and Parameters, and Their Nominal Values

Variable	Description	Nominal Value
a	Interfacial mass-transfer area/unit liquid holdup volume (m^2/m^3)	1,000
c_p	Molar heat capacity (J/mol·K)	80.0
E_a	Activation energy in Arrhenius rate expression (J/mol)	1.1×10^5
F_A	Inlet molar flow rate of reactant A (mol/s)	144.42
F_B	Inlet molar flow rate of reactant B (mol/s)	92.55
F_{1l}	Molar flow rate of liquid stream from reactor (mol/s)	110.97
F_{2l}	Molar flow rate of liquid recycle from condenser to reactor (mol/s)	66.45
F_{1v}	Molar flow rate of vapor stream from reactor to condenser (mol/s)	192.45
F_{2v}	Molar flow rate of product vapor from condenser (mol/s)	141.84
k_{10}	Preexponential factor (m^3/mol·s)	2.88×10^{11}
M_1^l	Liquid molar holdup in reactor (mol)	14,520
M_2^l	Liquid molar holdup in condenser (mol)	15,000
M_1^v	Vapor molar holdup in reactor (mol)	3,740.2
M_2^v	Vapor molar holdup in condenser (mol)	3,906.6
P_1	Pressure in reactor (Pa)	5.05×10^6
P_2	Pressure in condenser (Pa)	4.94×10^6
P^*	Setpoint for reactor pressure (Pa)	5.05×10^6
Q_1	Heat output from reactor (W)	8.6368×10^5
Q_2	Heat output from condenser (W)	1.0623×10^6
T_A	Temperature of feed A (K)	269.5
T_B	Temperature of feed B (K)	300.0
T_1	Temperature in reactor (K)	330.0
T_2	Temperature in condenser (K)	304.16
V_{1T}	Volume of reactor (m^3)	3.0
V_{2T}	Volume of condenser (m^3)	3.0
x_{A1}	Mole fraction of A in liquid phase in reactor	0.4937
x_{B1}	Mole fraction of B in liquid phase in reactor	0.4009
x_{A2}	Mole fraction of A in liquid phase in condenser	0.7355
y_{A1}	Mole fraction of A in vapor phase in reactor	0.47
y_{A2}	Mole fraction of A in vapor phase in condenser	0.33
ΔH_r	Heat of reaction (J/mol)	−50,000
ΔH^v	Latent heat of vaporization (J/mol)	10,000
ρ	Liquid molar density (mol/m^3)	15,000
R	Universal gas constant (J/mol·K)	8.314

$$0 = P_2 y_{A2} - P_{A2}^s x_{A2} \tag{82}$$

$$0 = P_2(1 - y_{A2}) - P_{C2}^s(1 - x_{A2}) \tag{83}$$

$$0 = P_1 - P^* \tag{84}$$

$$0 = P_1\left(V_{1T} - \frac{M_1^l}{\rho}\right) - M_1^v R T_1 \tag{85}$$

$$0 = P_2\left(V_{2T} - \frac{M_2^l}{\rho}\right) - M_2^v R T_2 \tag{86}$$

$$0 = P_1 - P_2 - \frac{1}{0.09}(F_{1v})^{7/4} \tag{87}$$

where P_{A1}^s, P_{A2}^s, P_{C1}^s, P_{C2}^s, and r_A are defined by

$$P_{A1}^s = \exp\left(25.1 - \frac{3{,}400}{T_1 + 20}\right) \tag{88}$$

$$P_{A2}^s = \exp\left(25.1 - \frac{3{,}400}{T_2 + 20}\right) \tag{89}$$

$$P_{C1}^s = \exp\left(27.3 - \frac{4{,}100}{T_1 + 70}\right) \tag{90}$$

$$P_{C2}^s = \exp\left(27.3 - \frac{4{,}100}{T_2 + 70}\right) \tag{91}$$

$$r_A = k_{10}\,\rho M_1^l x_{A1} x_{B1} \exp\left(-\frac{E_a}{RT_1}\right) \tag{92}$$

In this process, state variables are $x^T = [M_1^l, x_{A1}, x_{B1} M_1^v, y_{A1}, T_1, M_2^l, x_{A2}, M_2^v, y_{A2}, T_2]$; algebraic variables are $z^T = [N_{A1}, N_{C1}, N_{A2}, N_{C2}, F_{1v}, F_{2v}, P_1, P_2]$; the manipulated inputs are $u^T = [Q_1, Q_2, F_{2l}]$; the controlled outputs are $y^T = [T_1, y_{A2}, M_2^l]$. A description and nominal value of the process variables and parameters is included in Table 1.

According to the regularization algorithm, G_1^1 and G_2^1 can be selected as

$$G_1^1 = \begin{bmatrix} P_1 - P^* \\ P_2\left(V_{2T} - \frac{M_2^l}{\rho}\right) - M_2^v R T_2 \\ P_1 - P_2 - \frac{1}{0.09}(F_{1v})^{7/4} \end{bmatrix},$$

$$G_2^1 = \begin{bmatrix} P_1 y_{A1} - P_{A1}^2 x_{A1} \\ P_1(1 - y_{A1}) - P_{C1}^s(1 - x_{A1} - x_{B1}) \\ P_2 y_{A2} - P_{A2}^s x_{A2} \\ P_2(1 - y_{A2}) - P_{C2}^s(1 - x_{A2}) \\ P_1\left(V_{1T} - \frac{M_1^l}{\rho}\right) - M_1^v R T_1 \end{bmatrix}$$

A straightforward, but tedious, calculation shows that $G_1^2 = [g_1^2, \ldots, g_5^2]'$ is given by

$$g_1^2 = \Phi_{11} N_{A1} + \Phi_{12} N_{C1} + \Psi_{11} Q_1 + \Psi_{13} F_{2l} + \Gamma_1 \tag{93}$$

$$g_2^2 = \Phi_{21} N_{A1} + \Phi_{22} N_{C1} + \Psi_{21} Q_1 + \Psi_{23} F_{2l} + \Gamma_2 \tag{94}$$

$$g_3^2 = \Phi_{33} N_{A2} + \Phi_{34} N_{C2} + \Phi_{35} F_{1v} + \Phi_{36} F_{2v} + \Psi_{32} Q_2 + \Psi_{33} F_{2l} + \Gamma_3 \tag{95}$$

$$g_4^2 = \Phi_{43} N_{A2} + \Phi_{44} N_{C2} + \Phi_{45} F_{1v} + \Phi_{46} F_{2v} + \Psi_{42} Q_2 + \Psi_{43} F_{2l} + \Gamma_4 \tag{96}$$

$$g_5^2 = \Phi_{51} N_{A1} + \Phi_{52} N_{C1} + \Phi_{55} F_{1v} + \Psi_{51} Q_1 + \Psi_{53} F_{2l} + \Gamma_5 \tag{97}$$

where Φ_{ij}, Ψ_{ij}, and Γ_j are defined in Appendix C. In addition, $G = \begin{bmatrix} G_1^1 \\ G_1^2 \end{bmatrix}$, $G_1 = [(G_1^1)^T, g_1^2, \ldots, g_4^2]^T$ and $G_2 = g_5^2$. It is checked that $[(\partial G/\partial z)\,(\partial G/\partial u)]$ has full row rank, which implies that the regularization algorithm terminates at Step 2. It can be checked that (A2) and (A3) are satisfied.

The application of Algorithm 2 to the outputs y_1, y_2, and y_3 produces $r_1 = r_2 = r_3 = 1$ and

$$H_1^2 = \Phi_{61} N_{A1} + \Phi_{62} N_{C1} + \Psi_{61} Q_1 + \Psi_{63} F_{2l} + \Gamma_6 \tag{98}$$

$$H_2^2 = \Phi_{73} N_{A2} + \Phi_{74} N_{C2} + \Phi_{75} F_{1v} + \Gamma_7 \tag{99}$$

$$H_3^2 = \Phi_{83} N_{A2} + \Phi_{84} N_{C2} + \Psi_{83} F_{2l} + \Gamma_8 \tag{100}$$

where Φ_{ij}, Ψ_{ij}, and Γ_j are defined in Appendix C.

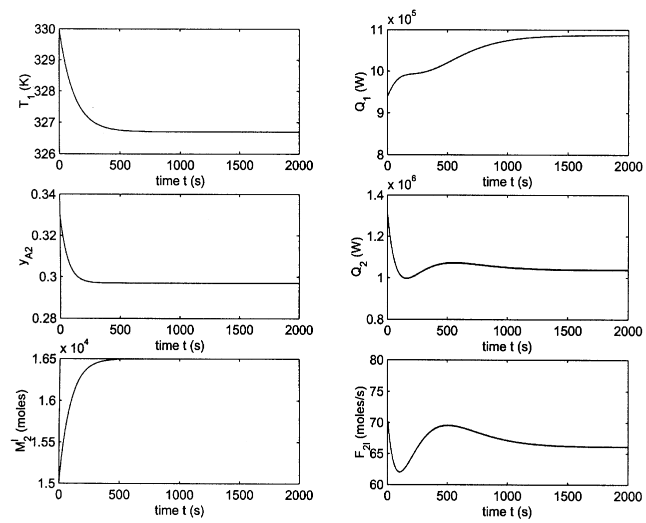

Figure 1. Current closed-loop profiles of controlled outputs and manipulated inputs corresponding to the controller: step signal tracking.

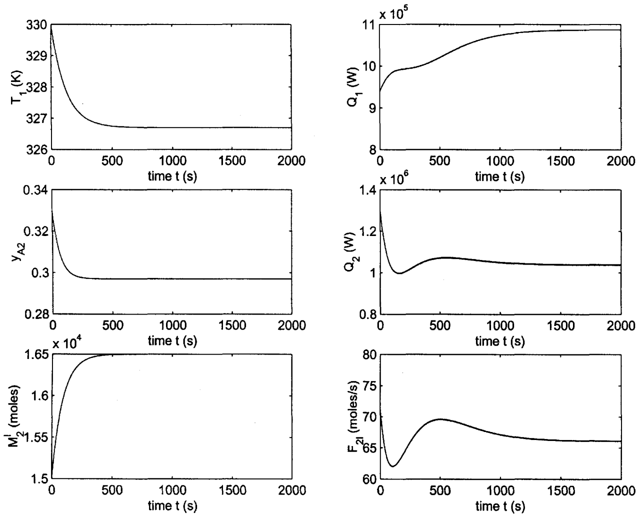

Figure 2. Closed-loop profiles of controlled outputs and manipulated inputs corresponding to the controllers in Kumar and Daoutidis (1996); step signal tracking.

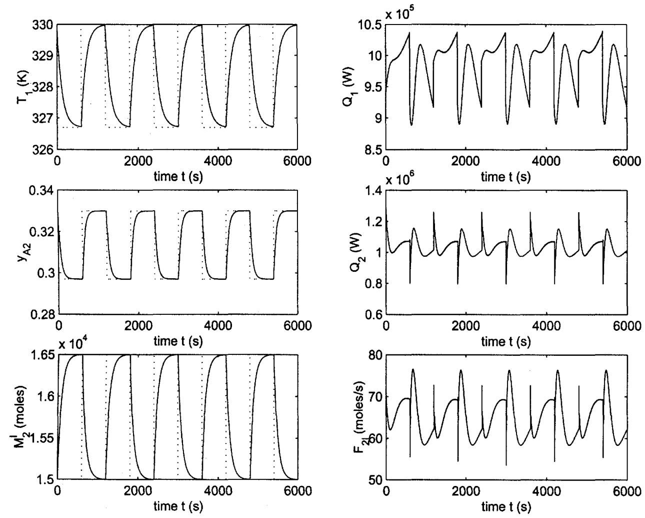

Figure 4. Closed-loop profiles of controlled outputs and manipulated inputs corresponding to the controllers in Kumar and Daoutidis (1996): square wave signal tracking.

Dashed line: set points; solid line: outputs.

Following the design procedure proposed in last section, it is derived that the tracking feedback controllers are given by

$$\dot{w} = v_1 \tag{101}$$

$$G_2 = M_2^v + w \tag{102}$$

$$G_1 = 0 \tag{103}$$

$$H_1^2 = -d_1\big(y_1 - y_1^d(t)\big) + \dot{y}_1^d(t) \tag{104}$$

$$H_2^2 = -d_2\big(y_2 - y_2^d(t)\big) + \dot{y}_2^d(t) \tag{105}$$

$$H_3^2 = -d_3\big(y_3 - y_3^d(t)\big) + \dot{y}_3^d(t) \tag{106}$$

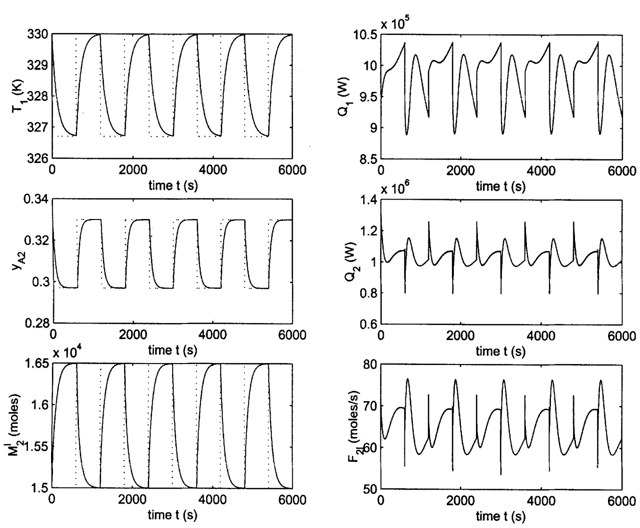

Figure 3. Current closed-loop profiles of controlled outputs and manipulated inputs corresponding to the controller: square wave signal tracking.

Dashed line: set points; solid line: outputs.

Two sets of simulation run are carried out with $d_1 = 1/120$, $d_2 = 1/60$, $d_3 = 1/90$ using both controllers in this article and in Kumar and Daoutidis (1996).

The first simulation run addressed the set point tracking capabilities of the proposed controller in the nominal case. For the process that was initially at its nominal steady state, a 1% decrease in the set point for the reactor temperature T_2, a 10% decrease in the set point for product composition y_{A2}, and 1% decrease in the set point for the liquid molar holdup M_2^l was imposed at $t = 0$. The closed-loop profiles of the three controlled outputs and manipulated inputs are shown in Figure 1. We can see that satisfactory tracking performance is achieved by the proposed controller. The tracking controllers based on the method in Kumar and Daoutidis (1996) are also simulated and the results are shown in Figure 2. It is clear that both controllers produce exactly the same simulation results.

The second set of the simulation run is carried out with square waves in the set points, with a period of 600 s. The simulation run addressed the time-varying signal tracking capabilities of the proposed controller in the nominal case. The closed-loop profiles of the three controlled outputs and manipulated inputs are shown in Figure 3. The satisfactory tracking performance is achieved by the proposed controller. The tracking controllers in Kumar and Daoutidis (1996) are also simulated under the same setting, and the simulation results are shown in Figure 4. It is clear that both controllers produce exactly the same simulation results.

Conclusion

This article has addressed the output tracking problem of nonlinear high-index DAE systems of the form (Eqs. 1–2). A two step design approach has been proposed to tackle the given DAE systems directly without finding the equivalent state-space representation. The first step is to carry out the regularization algorithm to test if the given system can be put into an index-one system. Then, another algorithm is performed to design a controller which achieves the tracking capabilities of the closed-loop system. The simulations performed on a chemical process comprised of a two-phase reactor and condenser illustrate the satisfactory control performance of the proposed design method.

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Appendix A: Proof of the Regularity of the Closed-Loop System (Eqs. 26–32)

In order to prove that the closed-loop system (Eqs. 26–32) is regular, z_1, z_2, u_1, and u_2 need to be determined first, then substitute them into the constraints to determine the constrained state-space region, and finally verify that the constrained state-space region is independent of the new inputs v_1 and v_2.

(a) Determine z_1 from Eq. 29: Since $\partial G_1/\partial z_1$ is nonsingular, it follows from the implicit function theorem that there exists a unique smooth function $z_1 = Z_1(x,z_2,u_1,u_2)$ which solves Eq. 29 locally.

(b) Determine z_2 from Eq. 32: Substituting $z_1 = Z_1(x,z_2,u_1,u_2)$ into Eq. 29 and differentiating it with respect to z_2 yields

$$\frac{\partial G_1}{\partial z_1}\frac{\partial Z_1}{\partial z_2} + \frac{\partial G_1}{\partial z_2} = 0$$

which implies that $\partial Z_1/\partial z_2 = -(\partial G_1/\partial z_1)^{-1}(\partial G_1/\partial z_2)$. According to the nonsingularities of the matrices $\partial G_1/\partial z_1$ and

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} \\ \dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial z_2} \end{bmatrix}$$

it follows from the relation

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} \\ \dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial z_2} \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} I & -\dfrac{\partial f_1}{\partial z_1}\left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1} \\ 0 & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1} \end{bmatrix}\begin{bmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} \\ \dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial z_2} \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} 0 & -\dfrac{\partial f_1}{\partial z_1}\left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial z_2} + \dfrac{\partial f_1}{\partial z_2} \\ I & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial z_2} \end{bmatrix} \end{aligned}$$

that $-(\partial f_1/\partial z_1)(\partial G_1/\partial z_1)^{-1}(\partial G_1/\partial z_2) + (\partial f_1/\partial z_2)$ is nonsingular. Let $\bar{f}_1(x,z_2,u_1,u_2) = f_1(x,Z_1(x,z_2,u_1,u_2),z_2,u_1,u_2)$. Then

$$\frac{\partial \bar{f}_1}{\partial z_2} = \frac{\partial f_1}{\partial z_1}\frac{\partial Z_1}{\partial z_2} + \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_1}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} + \frac{\partial f_1}{\partial z_2}$$

is nonsingular, which, according to the implicit function theorem, implies that there exists a unique smooth function $z_2 = Z_2(x,u_1,u_2,v_1)$, which, together with $z_1 = Z_1(x,z_2,u_1,u_2)$, solves Eq. 32 locally. In addition, the algebraic equation (Eq. 32) becomes $0 = \bar{f}_1(x,Z_2(x,u_1,u_2,v_1),u_1,u_2) + v_1$. Differentiating this last equation with respect to v_1 gives $(\partial \bar{f}_1/\partial z_2)$

$(\partial Z_2/\partial v_1) + I = 0$, that is

$$\frac{\partial Z_2}{\partial v_1} = \left(\frac{\partial \bar{f}_1}{\partial z_2}\right)^{-1} \tag{A1}$$

(c) Determine u_1 from Eq. 24: It follows from

$$s_1 = \text{rank}\left[\frac{\partial G_1}{\partial z_1}\right] = \text{rank}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} \left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & 0 \\ -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & I \end{bmatrix}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} \\ 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2} \end{bmatrix}$$

that $-(\partial G_2/\partial z_1)(\partial G_1/\partial z_1)^{-1}(\partial G_1/\partial z_2) + (\partial G_2/\partial z_2) = 0$, Let $\overline{G}_2(x,z_2,u_1,u_2) = G_2(x,Z_1(x,z_2,u_1,u_2),z_2,u_1,u_2)$. Then

$$\frac{\partial \overline{G}_2}{\partial z_2} = \frac{\partial G_2}{\partial z_1}\frac{\partial Z_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2}$$

$$= -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2} = 0$$

which implies that $\overline{G}_2(x,z_2,u_1,u_2)$ is independent of z_2. As a result, Eq. 24 becomes

$$\overline{G}_2(x,u_1,u_2) = x_1 + w \tag{A2}$$

Furthermore, according to the relation

$$\text{rank}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} \left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & 0 \\ -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & I \end{bmatrix}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} \\ 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} + \frac{\partial G_2}{\partial z_2} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} + \frac{\partial G_2}{\partial u_1} \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} \\ 0 & 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} + \frac{\partial G_2}{\partial u_1} \end{bmatrix}$$

it follows from the full row rankness of the matrix on the lefthand side of the above equation that the matrix $-(\partial G_2/\partial z_1)(\partial G_1/\partial z_1)^{-1}(\partial G_1/\partial u_1)+(\partial G_2/\partial u_1)$ is nonsingular, which implies that $\partial \overline{G}_2/\partial u_1$ is nonsingular. Thus, it follows from the implicit function theorem that there exists a unique smooth function $u_1 = U_1(x,u_2,w)$, which solves Eq. A2 locally.

(d) Determine u_2 from Eq. 25: $u_2 = v_2$.

In summary, u_1, u_2, z_1, and z_2 can be determined uniquely in terms of x, w, v_1, and v_2 from Eqs. 24, 25, 29, and 32.

(e) Determine the constrained state-space region: The constrained state-space region is defined by $x_1 + w = 0$, which is Eq. 30, and

$$\tilde{G}\big(x,Z_1\big(x,Z_2\big(x,U_1(x,v_2,w),v_2,v_1\big),U_1(x,v_2,w),v_2\big),$$
$$Z_2\big(x,U_1(x,v_2,w),v_2,v_1\big),U_1(x,v_2,w),v_2\big) = 0$$

which is obtained by substituting $z_1 = Z_1(x,z_2,u_1,u_2)$, $z_2 = Z_2(x,u_1,u_2,v_1)$, $u_1 = U_1(x,u_2,w)$, and $u_2 = v_2$ into $\tilde{G}(x,z_1,z_2,u_1,u_2) = 0$. Next, we will prove that it is independent of v_1 and v_2. First, according to the relation

$$\operatorname{rank}\begin{bmatrix} \frac{\partial \tilde{G}}{\partial z_1} & \frac{\partial \tilde{G}}{\partial z_2} & \frac{\partial \tilde{G}}{\partial u_1} & \frac{\partial \tilde{G}}{\partial u_2} \\ \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} I & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & 0 \\ 0 & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & 0 \\ 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & I \end{bmatrix}\begin{bmatrix} \frac{\partial \tilde{G}}{\partial z_1} & \frac{\partial \tilde{G}}{\partial z_2} & \frac{\partial \tilde{G}}{\partial u_1} & \frac{\partial \tilde{G}}{\partial u_2} \\ \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2}+\frac{\partial \tilde{G}}{\partial z_2} & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial \tilde{G}}{\partial u_1} & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial \tilde{G}}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2}+\frac{\partial G_2}{\partial z_2} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial G_2}{\partial u_1} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial G_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2}+\frac{\partial \tilde{G}}{\partial z_2} & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial \tilde{G}}{\partial u_1} & -\frac{\partial \tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial \tilde{G}}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial G_2}{\partial u_1} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial G_2}{\partial u_2} \end{bmatrix} \quad \text{(A3)}$$

it follows that $-(\partial\tilde{G}/\partial z_1)(\partial G_1/\partial z_1)^{-1}(\partial G_1/\partial z_2)+(\partial\tilde{G}/\partial z_2)=0$, Let $\overline{G}(x,z_2,u_1,u_2)=\tilde{G}(x,Z_1(x,z_2,u_1,u_2),z_2,u_1,u_2)$. Then

$$\frac{\partial\overline{G}}{\partial z_2}=\frac{\partial\tilde{G}}{\partial z_1}\frac{\partial Z_1}{\partial z_2}+\frac{\partial\tilde{G}}{\partial z_2}=-\frac{\partial\tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2}+\frac{\partial\tilde{G}}{\partial z_2}=0$$

which implies that $\overline{G}(x,z_2,u_1,u_2)$ is independent of z_2, that is, the algebraic equation $\tilde{G}(x,z_1,z_2,u_1,u_2)=0$ becomes $\overline{G}(x,u_1,u_2)=0$. At the same time, Eq. A3 becomes

$$\operatorname{rank}\begin{bmatrix}\frac{\partial\tilde{G}}{\partial z_1} & \frac{\partial\tilde{G}}{\partial z_2} & \frac{\partial\tilde{G}}{\partial u_1} & \frac{\partial\tilde{G}}{\partial u_2}\\ \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2}\\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2}\\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2}\end{bmatrix}$$

$$=\operatorname{rank}\begin{bmatrix}0 & 0 & -\frac{\partial\tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial\tilde{G}}{\partial u_1} & -\frac{\partial\tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial\tilde{G}}{\partial u_2}\\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}\\ 0 & 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial G_2}{\partial u_1} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial G_2}{\partial u_2}\end{bmatrix}\quad\text{(A4)}$$

which means that

$$\operatorname{rank}\begin{bmatrix}-\frac{\partial\tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial\tilde{G}}{\partial u_1} & -\frac{\partial\tilde{G}}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial\tilde{G}}{\partial u_2}\\ -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial G_2}{\partial u_1} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial G_2}{\partial u_2}\end{bmatrix}$$

$$=\operatorname{rank}\begin{bmatrix}-\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial G_2}{\partial u_1} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial G_2}{\partial u_2}\end{bmatrix}\quad\text{(A5)}$$

that is

$$\operatorname{rank}\begin{bmatrix}\frac{\partial\overline{G}}{\partial u_1} & \frac{\partial\overline{G}}{\partial u_2}\\ \frac{\partial\overline{G}_2}{\partial u_1} & \frac{\partial\overline{G}_2}{\partial u_2}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\frac{\partial\overline{G}_2}{\partial u_1} & \frac{\partial\overline{G}_2}{\partial u_2}\end{bmatrix}$$

where $\partial\overline{G}_2/\partial u_1$ is nonsingular, see (c) above. Substituting $u_1=U_1(x,u_2,w)$ into Eq. A2 and differentiating it with respect to u_2 gives

$$\frac{\partial\overline{G}_2}{\partial u_1}\frac{\partial U_1}{\partial u_1}+\frac{\partial\overline{G}_2}{\partial u_2}=0$$

which implies that $(\partial U_1/\partial u_1)=(\partial\overline{G}_2/\partial u_1)^{-1}(\partial\overline{G}_2/\partial u_1)$. Now let $\hat{G}(x,u_2)=\overline{G}(x,U_1(x,u_2),u_2)$. Then

$$\frac{\partial\hat{G}}{\partial u_2}=\frac{\partial\overline{G}}{\partial u_1}\frac{\partial U_1}{\partial u_1}+\frac{\partial\overline{G}}{\partial u_2}=-\frac{\partial\overline{G}}{\partial u_1}\left(\frac{\partial\overline{G}_2}{\partial u_1}\right)^{-1}\frac{\partial\overline{G}_2}{\partial u_1}+\frac{\partial\overline{G}}{\partial u_2}=0$$

because of

$$\operatorname{rank}\begin{bmatrix}\frac{\partial\overline{G}}{\partial u_1} & \frac{\partial\overline{G}}{\partial u_2}\\ \frac{\partial\overline{G}_2}{\partial u_1} & \frac{\partial\overline{G}_2}{\partial u_2}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\frac{\partial\overline{G}_2}{\partial u_1} & \frac{\partial\overline{G}_2}{\partial u_2}\end{bmatrix}$$

$$=\operatorname{rank}\begin{bmatrix}I & -\frac{\partial\overline{G}}{\partial u_1}\left(\frac{\partial\overline{G}_2}{\partial u_1}\right)^{-1}\\ 0 & \left(\frac{\partial\overline{G}_2}{\partial u_1}\right)^{-1}\end{bmatrix}\begin{bmatrix}\frac{\partial\overline{G}}{\partial u_1} & \frac{\partial\overline{G}}{\partial u_2}\\ \frac{\partial\overline{G}_2}{\partial u_1} & \frac{\partial\overline{G}_2}{\partial u_2}\end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} 0 & -\frac{\partial \overline{G}}{\partial u_1}\left(\frac{\partial \overline{G}_2}{\partial u_1}\right)^{-1}\frac{\partial \overline{G}_2}{\partial u_2}+\frac{\partial \overline{G}}{\partial u_2} \\ I & \left(\frac{\partial \overline{G}_2}{\partial u_1}\right)^{-1}\frac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

Therefore, $\hat{G}_2(x,u_2)$ is independent of u_2. So the algebraic equation

$$\tilde{G}\big(x,Z_1(x,Z_2(x,U_1(x,v_2,w),v_2,v_1),U_1(x,v_2,w),v_2),$$
$$Z_2(x,U_1(x,v_2,w),v_2,v_1),U_1(x,v_2,w),v_2\big)=0$$

is independent of v_1 and v_2, that is, the constrained state-space region is independent of v_1 and v_2. Thus, the closed-loop system (Eqs. 26–32) is regular.

Appendix B: Derivation of the Tracking Controller

It follows from Appendix A that $u_1 = U_1(x,u_2,w)$ and $u_2 = v_2$. So, in order to find the tracking controller for u_1 and u_2, we need to determine v_1 and v_2 from Eq. 60. To this end, let

$$H = \left[H_1^{r_1+1}, \ldots, H_m^{r_m+1}\right]^T$$

$$\tilde{H} = \left[H_1^1, \ldots, H_1^{r_1}; \ldots; H_m^1, \ldots, H_m^{r_m}\right]^T$$

$$Y(t) = \Big[y_1^d(t),(y_1^d)^{(1)}(t), \ldots,(y_1^d)^{(r_1)}(t); \ldots; y_m^d(t)$$
$$(y_m^d)^{(1)}(t), \ldots,(y_m^d)^{(r_m)}(t)\Big]^T$$

Then, Eq. 60 can be rewritten as

$$H(x,z_1,z_2,u_1,u_2) = M\big(\tilde{H}(x,z_1,z_2,u_1,u_2),Y(t)\big) \quad \text{(B1)}$$

Now, it follows from Algorithm 2 that

$$\text{rank}\begin{bmatrix} \frac{\partial \tilde{H}}{\partial z_1} & \frac{\partial \tilde{H}}{\partial z_2} & \frac{\partial \tilde{H}}{\partial u_1} & \frac{\partial \tilde{H}}{\partial u_2} \\ \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix}$$
$$= \text{rank}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix} \quad \text{(B2)}$$

and the matrix

$$\text{rank}\begin{bmatrix} \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \\ \frac{\partial H}{\partial z_1} & \frac{\partial H}{\partial z_2} & \frac{\partial H}{\partial u_1} & \frac{\partial H}{\partial u_2} \end{bmatrix}$$

is nonsingular due to (A4) in the Feedback Design Section.

By the same token as in (e) of Appendix A, it follows from Eq. B2 that the function

$$\tilde{H}\big(x,Z_1(x,Z_2(x,U_1(x,v_2,w),v_2,v_1),U_1(x,v_2,w),v_2)$$
$$Z_2(x,U_1(x,v_2,w),v_2,v_1),U_1(x,v_2,w),v_2\big)$$

is independent of v_1 and v_2.

Set

$$\overline{H}(x,z_2,u_1,u_2) = H[x,Z_1(x,z_2,u_1,u_2),z_2,u_1,u_2]$$

$$\hat{H}(x,u_1,u_2,v_1) = \overline{H}[x,Z_2(x,u_1,u_2,v_1),u_1,u_2]$$

$$H^1(x,w,v_1,v_2) = \hat{H}[x,U_1(x,v_2,w),v_1,v_2] \quad \text{(B3)}$$

Then

$$\frac{\partial \overline{H}}{\partial X} = \frac{\partial H}{\partial z_1}\frac{\partial Z_1}{\partial X} + \frac{\partial H}{\partial X} = -\frac{\partial H}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial X} + \frac{\partial H}{\partial X} = 0$$

where X represents one of z_2, u_1, and u_2. As a result, we have

---

$$\text{rank}\begin{bmatrix} \frac{\partial H}{\partial z_1} & \frac{\partial H}{\partial z_2} & \frac{\partial H}{\partial u_1} & \frac{\partial H}{\partial u_2} \\ \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} I & -\frac{\partial H}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & 0 \\ 0 & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & 0 \\ 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1} & I \end{bmatrix}\begin{bmatrix} \frac{\partial H}{\partial z_1} & \frac{\partial H}{\partial z_2} & \frac{\partial H}{\partial u_1} & \frac{\partial H}{\partial u_2} \\ \frac{\partial G_1}{\partial z_1} & \frac{\partial G_1}{\partial z_2} & \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} \\ \frac{\partial G_2}{\partial z_1} & \frac{\partial G_2}{\partial z_2} & \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & -\frac{\partial H}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2}+\frac{\partial H}{\partial z_2} & -\frac{\partial H}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial H}{\partial u_1} & -\frac{\partial H}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial H}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2}+\frac{\partial G_2}{\partial z_2} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1}+\frac{\partial G_2}{\partial u_1} & -\frac{\partial G_2}{\partial z_1}\left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2}+\frac{\partial G_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & \frac{\partial \overline{H}}{\partial z_2} & \frac{\partial \overline{H}}{\partial u_1} & \frac{\partial \overline{H}}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & 0 & \frac{\partial \overline{G}_2}{\partial u_1} & \frac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & \frac{\partial \overline{H}}{\partial z_2} & \frac{\partial \overline{H}}{\partial u_1} & \frac{\partial \overline{H}}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & 0 & \frac{\partial \overline{G}_2}{\partial u_1} & \frac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}\begin{bmatrix} \frac{\partial Z_2}{\partial v_1} & \frac{\partial Z_2}{\partial u_1} & \frac{\partial Z_2}{\partial u_2} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & \frac{\partial \overline{H}}{\partial z_2}\frac{\partial Z_2}{\partial v_1} & \frac{\partial \overline{H}}{\partial z_2}\frac{\partial Z_2}{\partial u_1}+\frac{\partial \overline{H}}{\partial u_1} & \frac{\partial \overline{H}}{\partial z_2}\frac{\partial Z_2}{\partial u_2}+\frac{\partial \overline{H}}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & 0 & \frac{\partial \overline{G}_2}{\partial u_1} & \frac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & \frac{\partial \hat{H}}{\partial v_1} & \frac{\partial \hat{H}}{\partial u_1} & \frac{\partial \hat{H}}{\partial u_2} \\ I & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial z_2} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_1} & \left(\frac{\partial G_1}{\partial z_1}\right)^{-1}\frac{\partial G_1}{\partial u_2} \\ 0 & 0 & \frac{\partial \overline{G}_2}{\partial u_1} & \frac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} I & 0 & -\dfrac{\partial \hat{H}}{\partial u_1}\left(\dfrac{\partial \overline{G}_2}{\partial u_1}\right)^{-1} \\ 0 & I & 0 \\ 0 & 0 & \left(\dfrac{\partial \overline{G}_2}{\partial u_1}\right)^{-1} \end{bmatrix}\begin{bmatrix} 0 & \dfrac{\partial \hat{H}}{\partial v_1} & \dfrac{\partial \hat{H}}{\partial u_1} & \dfrac{\partial \hat{H}}{\partial u_2} \\ I & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial z_2} & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial u_1} & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial u_2} \\ 0 & 0 & \dfrac{\partial \overline{G}_2}{\partial u_1} & \dfrac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & \dfrac{\partial \hat{H}}{\partial v_1} & 0 & \dfrac{\partial \hat{H}}{\partial u_1}\left(\dfrac{\partial \overline{G}_2}{\partial u_1}\right)^{-1}\dfrac{\partial \overline{G}_2}{\partial u_2} + \dfrac{\partial \hat{H}}{\partial u_2} \\ I & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial z_2} & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial u_1} & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial u_2} \\ 0 & 0 & I & \left(\dfrac{\partial \overline{G}_2}{\partial u_1}\right)^{-1}\dfrac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & \dfrac{\partial H^1}{\partial v_1} & 0 & \dfrac{\partial H^1}{\partial v_2} \\ I & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial z_2} & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial u_1} & \left(\dfrac{\partial G_1}{\partial z_1}\right)^{-1}\dfrac{\partial G_1}{\partial u_2} \\ 0 & 0 & I & \left(\dfrac{\partial \overline{G}_2}{\partial u_1}\right)^{-1}\dfrac{\partial \overline{G}_2}{\partial u_2} \end{bmatrix}$$

which implies that the matrix $[(\partial H^1/\partial v_1)\ (\partial H^1/\partial v_2)]$ is nonsingular. Up to now, we have proved that Eq. 60 or Eq. B1 can be rewritten as

$$H^1(x,w,v_1,v_2) = M\left[\tilde{H}(x,w),Y(t)\right] \tag{B4}$$

It follows from the implicit function theorem that there exists unique smooth functions $v_1 = V_1(x,w)$ and $v_2 = V_2(x,w)$ so that $H^1[x,w,v_1 = V_1(x,w),\ v_2 = V_2(x,w)] = M[\tilde{H}(x,w),Y(t)]$.

Appendix C: Expressions for Φ_{ij}, Ψ_{ij}, and Γ_j

$$\Phi_{11} = -\frac{P_1(1-y_{A1})}{M_1^v} - \frac{3{,}400 P_{A1}^s \Delta H^v x_{A1}}{\left(M_1^l + M_1^v\right)c_p(T_1+20)^2} - \frac{P_{A1}^s(1-x_{A1})}{M_1^l}$$

$$\Phi_{12} = -\frac{P_1 y_{A1}}{M_1^v} + \frac{3{,}400 P_{A1}^2 \Delta H^v x_{A1}}{\left(M_1^l + M_1^v\right)c_p(T_1+20)^2} - \frac{P_{A1}^s x_{A1}}{M_1^l}$$

$$\Psi_{11} = \frac{3{,}400 P_{A1}^s x_{A1}}{\left(M_1^l + M_1^v\right)c_p(T_1+20)^2}$$

$$\Psi_{13} = -\frac{3{,}400 P_{A1}^s x_{A1}(T_2-T_1)}{\left(M_1^l + M_1^v\right)(T_1+20)^2} - \frac{P_{A1}^s(x_{A2}-x_{A1})}{M_1^l}$$

$$\Gamma_1 = \frac{P_1 F_A(1-y_{A1})}{M_1^v} - \frac{3{,}400 P_{A1}^s x_{A1}}{\left(M_1^l + M_1^v\right)(T_1+20)^2} \times \left[F_A(T_A-T_1) + F_B(T_B-T_1) - r_A\frac{\Delta H_r}{c_p}\right] + \frac{P_{A1}^s(F_B x_{A1} + r_A)}{M_1^l}$$

$$\Phi_{21} = \frac{P_1(1-y_{A1})}{M_1^v} - \frac{4{,}100 P_{C1}^s \Delta H^v(1-x_{A1}-x_{B1})}{\left(M_1^l + M_1^v\right)c_p(T_1+70)^2} + \frac{P_{C1}^s(1-x_{A1})}{M_1^l} - \frac{P_{C1}^s x_{B1}}{M_1^l}$$

$$\Phi_{22} = \frac{P_1 y_{A1}}{M_1^v} + \frac{4{,}100 P_{C1}^s \Delta H^v(1-x_{A1}-x_{B1})}{\left(M_1^l + M_1^v\right)c_p(T_1+70)^2} + \frac{P_{C1}^s x_{A1}}{M_1^l} + \frac{P_{C1}^s x_{B1}}{M_1^l}$$

$$\Psi_{21} = \frac{4{,}100 P_{C1}^s(1-x_{A1}-x_{B1})}{\left(M_1^l + M_1^v\right)c_p(T_1+70)^2}$$

$$\Psi_{23} = -\frac{4{,}100 P_{C1}^s(1-x_{A1}-x_{B1})(T_2-T_1)}{\left(M_1^l + M_1^v\right)(T_1+70)^2} + \frac{P_{C1}^s(x_{A2}-x_{A1})}{M_1^l} - \frac{P_{C1}^s x_{B1}}{M_1^l}$$

$$\Gamma_2 = -\frac{P_1 F_A(1-y_{A1})}{M_1^v} - \frac{4{,}100 P_{C1}^s(1-x_{A1}-x_{B1})}{\left(M_1^l+M_1^v\right)(T_1+70)^2}$$
$$\times\left[F_A(T_A-T_1)+F_B(T_B-T_1)-r_A\frac{\Delta H_r}{c_p}\right] - \frac{P_{C1}^s(F_B x_{A1}+r_A)}{M_1^l}$$
$$+\frac{P_{C1}^s\left[F_B(1-x_{B1})-r_A\right]}{M_1^l}$$

$$\Phi_{33} = \frac{P_2 y_{A2}}{\rho} - R y_{A2} T_2 - \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_2(1-y_{A2})}{M_2^v}$$
$$+\left[y_{A2} M_2^v R - \frac{3{,}400}{(T_2+20)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right) P_{A2}^s x_{A2}\right]\frac{\Delta H^v}{\left(M_2^l+M_2^v\right)c_p}$$
$$-\left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_{A2}^s(1-x_{A2})}{M_2^l}$$

$$\Phi_{34} = \frac{P_s y_{A2}}{\rho} - R y_{A2} T_2 + \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_2 y_{A2}}{M_2^v}$$
$$+\left[y_{A2} M_2^v R - \frac{3{,}400}{(T_2+20)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right) P_{A2}^s x_{A2}\right]\frac{\Delta H^v}{\left(M_2^l+M_2^v\right)c_p}$$
$$+\left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_{A2}^s x_{A2}}{M_2^l}$$

$$\Phi_{35} = R y_{A2} T_2 + \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_2(y_{A1}-y_{A2})}{M_2^v}$$
$$+\left[y_{A2} M_2^v R - \frac{3{,}400}{(T_2+20)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right) P_{A2}^s x_{A2}\right]\frac{T_1-T_2}{M_2^l+M_2^v}$$

$$\Phi_{36} = -R y_{A2} T_2$$

$$\Psi_{32} = -\left[y_{A2} M_2^v R - \frac{3{,}400}{(T_2+20)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right) P_{A2}^s x_{A2}\right]$$
$$\times\frac{1}{\left(M_2^l+M_2^v\right)c_p}$$

$$\Psi_{33} = -\frac{P_2 y_{A2}}{\rho}$$

$$\Gamma_3 = 0$$

$$\Phi_{43} = \frac{P_2(1-y_{A2})}{\rho} - R(1-y_{A2})T_2 + \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_2(1-y_{A2})}{M_2^v}$$
$$+\left[(1-y_{A2}) M_2^v R - \frac{4{,}100}{(T_2+70)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right)\right.$$
$$\left.\times P_{C2}^s(1-x_{A2})\right]\frac{\Delta H^v}{\left(M_2^l+M_2^v\right)c_p} + \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_{C2}^s(1-x_{A2})}{M_2^l}$$

$$\Phi_{44} = \frac{P_2(1-y_{A2})}{\rho} - R(1-y_{A2})T_2 - \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_2 y_{A2}}{M_2^v}$$
$$+\left[(1-y_{A2}) M_2^v R - \frac{4{,}100}{(T_2+70)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right)\right.$$
$$\left.\times P_{C2}^s(1-x_{A2})\right]\frac{\Delta H^v}{\left(M_2^l+M_2^v\right)c_p} - \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_{C2}^s x_{A2}}{M_2^l}$$

$$\Phi_{45} = R(1-y_{A2})T_2 - \left(V_{2T}-\frac{M_2^l}{\rho}\right)\frac{P_2(y_{A1}-y_{A2})}{M_2^v}$$
$$+\left[(1-y_{A2}) M_2^v R - \frac{4{,}100}{(T_2+70)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right)\right.$$
$$\left.\times P_{C2}^s(1-x_{A2})\right]\frac{T_1-T_2}{M_2^l+M_2^v}$$

$$\Phi_{46} = -R(1-y_{A2})T_2$$

$$\Psi_{42} = -\left[(1-y_{A2}) M_2^v R - \frac{4{,}100}{(T_2+70)^2}\left(V_{2T}-\frac{M_2^l}{\rho}\right)\right.$$
$$\left.\times P_{C2}^s(1-x_{A2})\right]\frac{1}{\left(M_2^l+M_2^v\right)c_p}$$

$$\Psi_{43} = -\frac{P_2(1-y_{A2})}{\rho}$$

$$\Gamma_3 = 0$$

$$\Phi_{51} = -\frac{P_1}{\rho} + RT_1 - \frac{RM_1^v \Delta H^v}{\left(M_1^l+M_1^v\right)c_p}$$

$$\Phi_{52} = \frac{P_1}{\rho} - RT_1 + \frac{RM_1^v \Delta H^v}{\left(M_1^l+M_1^v\right)c_p}$$

$$\Phi_{55} = RT_1$$

$$\Psi_{51} = \frac{RM_1^v}{\left(M_1^l+M_1^v\right)c_p}$$

$$\Psi_{53} = -\frac{P_1}{\rho} - \frac{RM_1^v(T_2-T_1)}{M_1^l+M_1^v}$$

$$\Gamma_5 = -\frac{P_1}{\rho}(F_B-F_{1l}) - RT_1F_A - \frac{RM_1^v}{M_1^l+M_1^v}$$
$$\times\left[F_A(T_A-T_1)+F_B(T_B-T_1)-r_A\frac{\Delta H_r}{c_p}\right]$$

$$\Phi_{61} = \frac{\Delta H^v}{\left(M_1^l+M_1^v\right)c_p}$$

$$\Phi_{62} = -\frac{\Delta H^v}{\left(M_1^l+M_1^v\right)c_p}$$

$$\Psi_{61} = -\frac{1}{\left(M_1^l+M_1^v\right)c_p}$$

$$\Psi_{63} = \frac{T_2 - T_1}{M_1^l + M_1^v}$$

$$\Gamma_6 = \frac{1}{M_1^l + M_1^v}\left[F_A(T_A - T_1) + F_B(T_B - T_1) - r_A \frac{\Delta H_r}{c_p}\right]$$

$$\Phi_{73} = -\frac{1 - y_{A2}}{M_2^v}$$

$$\Phi_{74} = \frac{y_{A2}}{M_2^v}$$

$$\Phi_{75} = \frac{y_{A1} - y_{A2}}{M_2^v}$$

$$\Gamma_7 = 0$$

$$\Phi_{83} = 1$$

$$\Phi_{84} = 1$$

$$\Psi_{83} = -1$$

$$\Gamma_8 = 0$$

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